

Chaotic System on the Super Riemann Surface

SHUJI MATSUMOTO[★]*KEK, National Laboratory for High Energy
Physics, Oho, Tsukuba, Ibaraki 305, Japan*SHOZO UEHARA[†]*Uji Research Center, Yukawa Institute for
Theoretical Physics, Kyoto University, Uji 611, Japan*

and

YUKINORI YASUI[‡]*Department of Physics, Osaka City
University, Sumiyoshi, Osaka 558, Japan*

ABSTRACT

A chaotic system of a nonrelativistic superparticle moving freely on the super Riemann surface (SRS) of genus $g \geq 2$ is reviewed.

[★] E-mail address: shuji@kekvox.kek.jp

[†] E-mail address: uehara@yisun1.yukawa.kyoto-u.ac.jp

[‡] E-mail address: f51999@sakura.kudpc.kyoto-u.ac.jp

In the present paper, we review a supersymmetric extension of the Hadamard model, the classical and quantum motions of a superparticle on the super Riemann surface (SRS).

The conventional Hadamard model [1] represents the free motion of a nonrelativistic particle on the compact Riemann surface of a constant negative curvature [2]. The classical motion is known for its strongly chaotic property. The Hadamard model was originally proposed as a simple example of the dynamical system which possesses the ergodic property. This model has various advantages in the study of the chaotic properties mathematically. One of the essential features of this model is that there exists a well-defined quantum dynamics where the Laplace-Beltrami operator on the Riemann surface acts as the Hamiltonian. The model may be useful in physics to examine the properties of a quantum chaotic system. Specifically, the model shows us the relation between a classical and a quantum chaos. A quantized energy sum rule is actually the celebrated Selberg trace formula [3, 4]. The quantum energy spectrum is complicated, however, it would be obtained through the Selberg zeta function.

Here, we intend to bring the supersymmetry into the Hadamard model. Then, we investigate the system of a superparticle moving freely on a compact super Riemann surface of genus $g \geq 2$. The supersymmetrical Hadamard model will offer

1. an application of the superanalog of the analytic theories on a Grassmann algebra [5].
2. an example of integrable classical and quantum dynamical systems with supersymmetry (the motion on SH, a universal covering space of the SRS, is expected to be integrable).
3. the notion of supersymmetrized chaos (however it seems to be rather puzzling).
4. superanalogues of the trace formula and the zeta function, which are important for the superstring theory (the notion of an SRS comes naturally from the superspace approach of superstrings [6]).

Motivated by them and armed with the mathematical tools for the supersymmetry, we develop the theory along the conventional study of the Hadamard model. This paper is organized as follows. In the next section the notations and the conventions of super Riemann surfaces are presented and the Lagrangian of a superparticle on a super Riemann surface is given. Section 3 is devoted to the classical mechanics for the system and quantization is carried out in Sec. 4. Superanalogs of the Selberg trace formula and the zeta function are given in Sec. 5. Section 6 is devoted to the discussions on the classical chaos. In the final section we comment on the moduli space of the super Riemann surface. This paper is based on our previous works [7 – 11].

2. Preliminaries

This section is devoted to compiling the notations and conventions of super Riemann surfaces (SRSs) of genus $g \geq 2$. As in the conventional Hadamard model, we will employ here the convenient way to represent the SRS. We will see that it is represented as a fundamental domain of a certain universal covering space. And this brings us an advantage that we can investigate the motions on an SRS by imposing a periodic boundary condition on the motions on the covering space. The free motion on an SRS is supposed to be generated by the *distance-proportional* Lagrangian. We will give a metric tensor on the covering space parametrized by a real parameter a and also give the Lagrangian for a superparticle based on the distance.

As is well known, the Poincaré upper half-plane $H = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ together with the group of its conformal automorphisms $\text{PSL}(2, \mathbb{R})$ (Möbius group) is a model of the hyperbolic geometry. Turning to the supergeometry, we can extend all the standard constructions to the superplanes. The super Poincaré upper half-plane SH is expressed by one Grassmann even and one odd complex coordinate z and θ , respectively;^{*}

$$\text{SH} = \{Z = (z, \theta) \mid \text{Im } z > 0\}. \quad (1)$$

^{*} $\text{Im } z > 0$ means that $\text{Im } z_0 > 0$ with z_0 being the body part of z . We shall use such a convention for inequalities through out this paper for simplicity.

The superconformal automorphisms $\text{SPL}(2, \mathbb{R})$ of SH consist of such transformation $k : (z, \theta) \mapsto (\tilde{z}, \tilde{\theta})$ as

$$\begin{aligned} \tilde{z} &= \frac{az + b}{cz + d} + \theta \frac{\alpha z + \beta}{(cz + d)^2}, \\ \tilde{\theta} &= \frac{\alpha z + \beta}{cz + d} + \theta \frac{1 + \frac{1}{2}\beta\alpha}{cz + d}, \end{aligned} \quad (2)$$

where a, b, c and d are Grassmann even and α and β are Grassmann odd parameters with[†]

$$ad - bc = 1, \quad a, b, c, d \in \mathbb{R}. \quad (3)$$

The above transformation (2) is, of course, superanalytic and is also superconformal,

$$D\tilde{z} - \tilde{\theta}D\tilde{\theta} = 0, \quad (4)$$

$$D \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}. \quad (5)$$

If we introduce homogeneous coordinates (z_1, z_2, ξ) of the complex projective space, we can rewrite (2) as a linear transformation $(z = z_1 z_2^{-1}, \theta = \xi z_2^{-1})$,

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{\xi} \end{pmatrix} = A_k \begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix}, \quad (6)$$

$$A_k = \left(1 + \frac{1}{2}\beta\alpha\right)^{-1} \times \begin{pmatrix} a & b & b\alpha - a\beta \\ c & d & d\alpha - c\beta \\ \alpha & \beta & 1 + \frac{3}{2}\beta\alpha \end{pmatrix}, \quad \text{sdet } A_k = 1.$$

Similarly to the ordinary Riemann surfaces, every compact super Riemann surface with genus $g \geq 2$ can be represented as a quotient space SH/ST [12], where th

[†] As to complex conjugation, we adopt such a convention that $\bar{\alpha} = i\alpha, \bar{\beta} = i\beta$.

universal covering space of a super Riemann surface (SRS) is the super Poincaré upper half-plane SH and the covering group $\text{S}\Gamma$ (called the super Fuchsian group) is a discrete subgroup of superconformal automorphisms $\text{SPL}(2, \mathbb{R})$ having no fixed points on SH.

The super Fuchsian group $\text{S}\Gamma$ is generated by $2g$ elements $\{A_i, B_i, i = 1, \dots, g\}$ satisfying a condition,

$$\prod_{i=1}^g (A_i B_i A_i^{-1} B_i^{-1}) = 1. \quad (7)$$

Each element of the generators contains three Grassmann even and two odd parameters and the condition (7) is invariant under $A_i \mapsto M A_i M^{-1}$, $B_i \mapsto M B_i M^{-1}$ where $M \in \text{SPL}(2, \mathbb{R})$. Thus the set of the generators and hence the SRS with genus $g \geq 2$ are specified by $6g - 6$ Grassmann even and $4g - 4$ odd parameters. $\text{S}\Gamma$ acts discontinuously on SH and all its elements are hyperbolic, i.e., the reduced subgroup, where odd parameters are put to be zero, consists of the hyperbolic elements, $|a + d| > 2$. Let $\text{Conj}(\text{S}\Gamma)$ be the set of all conjugacy classes of $\text{S}\Gamma$ and $\text{Prim}(\text{S}\Gamma) = \{p \in \text{Conj}(\text{S}\Gamma); p \neq k^m \text{ for any } k \in \text{Conj}(\text{S}\Gamma) \text{ and } m \geq 2\}$ the set of all primitive conjugacy classes of $\text{S}\Gamma$. Then we have

$$Q(\text{Prim}(\text{S}\Gamma)) = \text{Conj}(\text{S}\Gamma), \quad \text{where } Q(P) = \{p^m, p \in P, m \geq 0\}. \quad (8)$$

An element $k \neq 1$ of $\text{S}\Gamma$ causes such a transformation as (2). $\text{S}\Gamma$ acts effectively on SH, however, $k \in \text{S}\Gamma$ has two fixed points, (u, μ) and (v, ν) , on the “super” real axis $\mathbb{R}_s \equiv \{Z = (z, \theta) \mid \text{Im } z = 0, \bar{\theta} = i\theta\}$,

$$u, v = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}, \quad \mu = \frac{\alpha u + \beta}{cu + d - 1}, \quad \nu = \frac{\alpha v + \beta}{cv + d - 1}. \quad (9)$$

These fixed points are repelling and attractive points, respectively,

$$\begin{aligned} k^{-n} : (z, \theta) &\longrightarrow (u, \mu), \\ k^n : (z, \theta) &\longrightarrow (v, \nu), \quad \text{for } n \rightarrow \infty. \end{aligned} \quad (10)$$

Let us define the quantities N_k (norm function) and $\chi(k)$ (sign factor) for $k \in \text{S}\Gamma$

$$\chi(k)(N_k^{1/2} + N_k^{-1/2}) = a + d - \frac{a + d + 2}{2}\beta\alpha = \text{str } A_k + 1. \quad (11)$$

Actually, $N_k (> 1)$ is the square of the maximum eigenvalue of the matrix A_k and $\chi(k)$ has to be chosen as

$$\chi(k) = \begin{cases} 1, & \text{if } \text{str } A_k + 1 > 2; \\ -1, & \text{if } \text{str } A_k + 1 < -2, \end{cases} \quad (12)$$

Using the transformation of $\text{SPL}(2, \mathbb{R})$, we see that any element $k \neq 1$ of $\text{S}\Gamma$ always conjugate in $\text{SPL}(2, \mathbb{R})$ to the magnification

$$\begin{aligned} \tilde{w} &= N_k w, \\ \tilde{\eta} &= \chi(k) N_k^{1/2} \eta, \quad N_k > 1, \end{aligned} \quad (13)$$

or in the matrix representation:

$$A_f A_k A_f^{-1} = A_{f k f^{-1}} = \text{diag} \left(\chi(k) N_k^{1/2}, \chi(k) N_k^{-1/2}, 1 \right) \equiv A_{\text{mag}}. \quad (14)$$

Apparently, the magnification depends on the element of $\text{Conj}(\text{S}\Gamma)$,

$$N_{g k g^{-1}} = N_k, \quad \chi(g k g^{-1}) = \chi(k). \quad (15)$$

Now we will introduce a $\text{SPL}(2, \mathbb{R})$ -invariant metric tensor on SH which is superanalog of the Poincaré metric on H. The latter, $ds_0^2 = |dz|^2 / (\text{Im } z)^2$, is invariant under $\text{PSL}(2, \mathbb{R})$ and gives a constant negative curvature. The corresponding volume element is

$$\frac{dx dy}{y^2}, \quad (\text{Re } z = x, \text{Im } z = y). \quad (16)$$

To construct the $\text{SPL}(2, \mathbb{R})$ -invariant metric tensor on SH, we use the technique developed in the supergravity theory on $2 + 2$ dimensional curved superspace

The basic quantities are the super vielbein E_M^A which, however, are not completely independent superfields. It was shown that 2 + 2 dimensional superspace is superconformally flat [13] where the basis one-forms \hat{E}^A are

$$\hat{E}^{++} = dz + \theta d\theta, \quad \hat{E}^{--} = d\bar{z} - \bar{\theta} d\bar{\theta}, \quad \hat{E}^+ = d\theta, \quad \hat{E}^- = d\bar{\theta}. \quad (17)$$

By the super Weyl transformation [13] the vielbein \hat{E}_M^A changes as

$$\hat{E}_M^A \mapsto E_M^A = \begin{cases} E_M^a = \Lambda(Z) \hat{E}_M^a, \\ E_M^\alpha = \Lambda^{1/2}(Z) \hat{E}_M^\alpha - i \hat{E}_M^a (\gamma_a)^{\alpha\beta} \hat{E}_\beta^N \partial_N \Lambda^{1/2}(Z), \end{cases} \quad (18)$$

($a = ++, --, \alpha = +, -$)

where $\hat{E}_A^M = (\hat{E}_M^A)^{-1}$, $\Lambda(Z)$ is the scaling function and (γ_a) is the gamma matrix:

$$(\gamma_+)^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad (\gamma_-)^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (19)$$

We take a SPL(2, \mathbb{R})-covariant function for $\Lambda(Z)$,

$$\Lambda(Z) = Y^{-1}, \quad (20)$$

where Y is given by

$$Y = \text{Im } z + \frac{1}{2} \theta \bar{\theta}, \quad (21)$$

which is the superanalog of $y = \text{Im } z$ on H. The E^A are now given as

$$\begin{aligned} E^{++} &= Y^{-1} (dz + \theta d\theta), \quad E^{--} = \overline{E^{++}}, \\ E^+ &= Y^{-3/2} \left[\left(Y + \frac{1}{2} \theta \bar{\theta} \right) d\theta + \frac{1}{2} (i\theta - \bar{\theta}) dz \right], \quad E^- = \overline{E^+}. \end{aligned} \quad (22)$$

The SPL(2, \mathbb{R})-invariant line element can be constructed by

$$ds^2 = E^{++} E^{--} - 2a E^+ E^-, \quad (23)$$

where $a (\neq 0)$ is an arbitrary real Grassmann even number. Rewriting $ds^2 =$

$dq^A g_{AB} dq^B$, $(q^z, q^{\bar{z}}, q^\theta, q^{\bar{\theta}}) = (z, \bar{z}, \theta, \bar{\theta})$, we obtain the metric tensor on SH,

$$(g_{AB}) = \begin{pmatrix} 0 & \frac{1}{2Y^2} & 0 & -\frac{\bar{\theta} + a(i\theta - \bar{\theta})}{2Y^2} \\ \frac{1}{2Y^2} & 0 & \frac{\theta - a(\theta + i\bar{\theta})}{2Y^2} & 0 \\ 0 & \frac{a(\theta + i\bar{\theta}) - \theta}{2Y^2} & 0 & \frac{\theta \bar{\theta} - 2a(Y + \theta \bar{\theta})}{2Y^2} \\ \frac{\bar{\theta} + a(i\theta - \bar{\theta})}{2Y^2} & 0 & \frac{2a(Y + \theta \bar{\theta}) - \theta \bar{\theta}}{2Y^2} & 0 \end{pmatrix}, \quad (24)$$

and the corresponding volume element is given by

$$dV = \frac{1}{2aY} dx dy d\theta d\bar{\theta}. \quad (25)$$

Since we have now a SPL(2, \mathbb{R})-invariant line element (23), we give our Lagrangian of a superparticle with mass m on SH [9],

$$L = \frac{m}{2} \left(\frac{ds}{dt} \right)^2 = \frac{m}{2} \dot{q}^A g_{AB} \dot{q}^B. \quad (26)$$

This is SPL(2, \mathbb{R})-invariant and hence, of course, SF-invariant, thus it is also the Lagrangian for a superparticle on the SRS.

3. Classical Mechanics

In this section we examine the classical dynamics of a superparticle on the SRS. Firstly, we solve the motion on SH. The motion on SH is expected to be integrable. According to the canonical theory of supermechanics, if we find out the adequate number of integrals of motion which *commute* each other with respect to the Poisson bracket, we can construct the general solution out of the integrals. However it is rather difficult to do it explicitly. There is no systematical way to get such integrals, and hence we take another path. We solve the Hamilton-Jacobi equation. The calculation is considerably cumbersome but the general solution for the metric tensor with an arbitrary parameter a is given. Since the SRS is represented by the fundamental domain of SH, SH/SF, the motion on the SRS is given by imposing the *periodic* boundary condition on the motion on SH. As we have expected, the motion on the SRS shows chaotic properties.

The Euler-Lagrange equations from L in (26) are geodesic equations,

$$\ddot{q}^A + \Gamma_{BC}^A \dot{q}^C \dot{q}^B = 0, \quad (27)$$

where Γ_{BC}^A is the Cristoffel's symbol [11]. Eq.(27) is given explicitly,

$$\begin{aligned} \ddot{z} + \frac{1}{Y}(i\dot{z}^2 - \dot{z}\dot{\theta}\bar{\theta}) + \frac{1-a}{2a} \left(\frac{i}{Y^2} \theta\bar{\theta}\dot{z}\dot{z} - \frac{2}{Y} \dot{z}\theta\dot{\theta} \right) &= 0, \\ \ddot{\theta} + \frac{i}{Y} \dot{z}\dot{\theta} + \frac{1-a}{2a} \left(\frac{2Y + \theta\bar{\theta}}{Y^2} \dot{z}\dot{\theta} - \frac{\theta + i\bar{\theta}}{Y^2} \dot{z}\dot{z} + \frac{2}{Y} \theta\dot{\theta}\dot{\theta} + \frac{i}{Y^2} \dot{z}\theta\bar{\theta}\dot{\theta} \right) &= 0, \end{aligned} \quad (28)$$

and their complex conjugated ones. The body part of the Eqs.(28) is

$$\ddot{z}_0 + \frac{i}{y_0} \dot{z}_0^2 = 0, \quad (29)$$

which is the geodesic equation on \mathbb{H} with the Poincaré metric [1]. The solutions to (29) are give by^{*},

$$z_0(t) = c_1 \frac{\sinh X_0 + i}{\cosh X_0} + c_2, \quad \text{and} \quad i e^{X_0} + c_2, \quad (30)$$

where

$$X_0 \equiv \omega(t + t_0), \quad c_1, c_2, \omega, t_0 \in \mathbb{R}. \quad (31)$$

A classical motion is determined uniquely with the boundary conditions which are the position and the velocity at the initial point. Thus the constants of the integration for the Euler-Lagrange equation (27) or (28) are four real Grassmann even and also four odd constants. So expanding z and θ in the Grassmann odd

^{*} The second solution is in fact obtained by taking a proper limit of the first solution. The first solution is always transformed into the second one by an appropriate Möbius transformation.

constants, say, $\epsilon_1, \bar{\epsilon}_1, \epsilon_2, \bar{\epsilon}_2$, we have a set of differential equations for the coefficients of the Grassmann even functions. However, it is actually not easy to solve those equations. So instead of solving (28) directly, we will take a roundabout. The Hamilton-Jacobi equation is given by

$$\frac{\partial S}{\partial t} + \frac{1}{2m} g^{BA} \frac{\partial S}{\partial q^A} \frac{\partial S}{\partial q^B} = 0. \quad (32)$$

Since the action which the classical solutions are plugged into satisfies the above equation (32), we express S as

$$S(q_1, q_2; t_2 - t_1) = \frac{m}{2} \int_{t_1}^{t_2} dt \dot{q}^A(t) g_{AB}(q(t)) \dot{q}^B(t), \quad (33)$$

where $q^A(t)$ is a solution of the geodesic equation (27) connecting the initial point $q_1 = q(t_1)$ and the final point $q_2 = q(t_2)$. It can be easily shown that the integrand is time-independent and its body part is non-negative. Taking them into account we set

$$\dot{q}^A(t) g_{AB}(t) \dot{q}^B(t) = (\text{const.}) \equiv \omega^2, \quad (34)$$

and define a superanalog of the hyperbolic distance,

$$d(q_1, q_2) = \int_{t_1}^{t_2} dt \sqrt{\dot{q}^A(t) g_{AB}(t) \dot{q}^B(t)} = |\omega|(t_2 - t_1). \quad (35)$$

From (33), (34) and (35), we get

$$S(q_1, q_2; t) = \frac{m}{2} \frac{[d(q_1, q_2)]^2}{t}. \quad (36)$$

Note that the hyperbolic distance $d_0(q_1, q_2)$ between $((q_1)_0) = (z_0, \bar{z}_0)$ and $((q_2)_0) = (w_0, \bar{w}_0)$, which should be the body part of $d(q_1, q_2)$, is given by

$$\cosh d_0 = 1 + \frac{|z_0 - w_0|^2}{2 \text{Im } z_0 \text{Im } w_0} \equiv 1 + \frac{1}{2} R_0, \quad (37)$$

which is $\text{PSL}(2, \mathbb{R})$ -invariant. And hence $d(q_1, q_2)$ should be symmetric under the exchange of q_1 and q_2 and $\text{SPL}(2, \mathbb{R})$ -invariant. There exist two basic functions

on $\text{SH} \times \text{SH}$ with such properties [14],

$$R(q_1, q_2) = \frac{|z_1 - z_2 - \theta_1 \theta_2|^2}{Y_{(1)} Y_{(2)}} , \quad (38)$$

$$r(q_1, q_2) = \left\{ \frac{2\theta_1 \bar{\theta}_1 + i(\theta_2 - i\bar{\theta}_2)(\theta_1 + i\bar{\theta}_1)}{4Y_{(1)}} + (1 \leftrightarrow 2) \right\} + \frac{(\theta_2 + i\bar{\theta}_2)(\theta_1 + i\bar{\theta}_1) \text{Re}(z_1 - z_2 - \theta_1 \theta_2)}{4Y_{(1)} Y_{(2)}} , \quad (39)$$

where $Y_{(i)} = \text{Im } z_i + \frac{1}{2} \theta_i \bar{\theta}_i$ for $i = 1, 2$. Here R is the superanalog of R_0 in (37) and r is nilpotent, and hence we can expect that $d(q_1, q_2)$ takes in general the following form,

$$\cosh d = f(R) + k(R) r . \quad (40)$$

The Hamilton-Jacobi equation leads to the differential equations for the unknown functions f and k , and we find that the “super” hyperbolic distance $d(q_1, q_2)$ is given by [10],

$$\cosh[d(q_1, q_2)] = 1 + \frac{1}{2} R(q_1, q_2) + k(R) r(q_1, q_2), \quad (41)$$

where

$$k(R) = \cosh l - 1 - \sinh l \coth \frac{l}{2a} , \quad (42)$$

$$l = l(q_1, q_2) = \cosh^{-1} \left(1 + \frac{1}{2} R \right) .$$

The next step is to solve $q_1 \equiv q$ in terms of q_2 and its canonical conjugated quantity, say, $p^{(2)}$. This can be done by solving the following algebraic equations with respect to q ,

$$\frac{\partial S}{\partial q_2^A} = -p_A^{(2)} , \quad (43)$$

where q_2 's and $p^{(2)}$'s actually correspond to the constants of integration for the differential equations (28). The calculation is cumbersome but rather straightforward

and we obtain the solution of the Euler-Lagrange equations (28), $(z^{(\text{I})}(t), \theta^{(\text{I})}(t))$, $(z^{(\text{II})}(t), \theta^{(\text{II})}(t))$ or $(z^{(\text{III})}(t), \theta^{(\text{III})}(t))$, [10],

$$z^{(\text{I})}(t) = \left[c_1 - \frac{2}{\cosh X} \{ i\xi_1 \xi_2 e^{-\frac{X}{a}} - i\xi_3 \xi_4 e^{\frac{X}{a}} - \xi_1 \xi_4 e^{(1-\frac{1}{a})X} + \xi_2 \xi_3 e^{(\frac{1}{a}-1)X} \} \right] \frac{\sinh X + i}{\cosh X} + c_2 ,$$

$$\theta^{(\text{I})}(t) = \left(\frac{\sinh X + i}{\cosh X} + 1 \right) \{ \xi_1 e^{-\frac{X}{a}} - i\xi_2 e^{-X} + i\xi_3 e^{(\frac{1}{a}-1)X} + \xi_4 \} , \quad (44)$$

$$z^{(\text{II})}(t) = i e^X + c_2 + i\xi_1 \xi_2 e^{(1-\frac{1}{a})X} - i\xi_3 \xi_4 e^{(1+\frac{1}{a})X} - \xi_1 \xi_4 e^{(2-\frac{1}{a})X} + \xi_2 \xi_3 e^{\frac{X}{a}} ,$$

$$\theta^{(\text{II})}(t) = i\xi_1 e^{(1-\frac{1}{a})X} + \xi_2 - \xi_3 e^{\frac{X}{a}} + i\xi_4 e^X , \quad (45)$$

$$z^{(\text{III})}(t) = i c_1 + c_2 - 2 a c_1 \omega_s t - \omega_s \{ 2 i a^2 c_1 \omega_s - a \bar{\epsilon}_2 \epsilon_1 + (1-a) \epsilon_2 \bar{\epsilon}_1 \} t^2 - \frac{1}{3} \epsilon_1 \bar{\epsilon}_1 \omega_s t^3 ,$$

$$\theta^{(\text{III})}(t) = \epsilon_2 + \epsilon_1 t + \left[\{ i a \epsilon_1 + (1-a) \bar{\epsilon}_1 \} \omega_s - \frac{1-a}{2 a c_1} \epsilon_1 \bar{\epsilon}_1 \epsilon_2 \right] t^2 , \quad (46)$$

where

$$X \equiv \omega(t + t_0) ,$$

$$\omega, t_0, c_1, c_2 : \text{ real Grassmann even constants, } c_1 > 0,$$

$$\xi_k \ (k = 1, 2, 3, 4) : \text{ Grassmann odd constants with } \bar{\xi}_k = i \xi_k , \quad (47)$$

$$\epsilon_1, \epsilon_2 : \text{ complex Grassmann odd constants,}$$

$$\omega_s : \text{ Grassmann even constant with no body part.}$$

The first $(z^{(\text{I})}, \theta^{(\text{I})})$ and the second $(z^{(\text{II})}, \theta^{(\text{II})})$ solutions correspond to the first and the second solutions in (30), respectively and the third one $(z^{(\text{III})}, \theta^{(\text{III})})$ corresponds to the solution with $\omega = 0$ in (30). And actually $(z^{(\text{II})}, \theta^{(\text{II})})$ is obtained by taking a proper limit of $(z^{(\text{I})}, \theta^{(\text{I})})$.

Now we examine the classical motion on the SRS. Since we have obtained the classical paths (44), (45) and (46) on the covering space SH of the SRS, we can deduce the classical motion on the SRS through projecting the paths on S

onto the fundamental domain $\text{SH}/\text{S}\Gamma$. We study closed orbits on the SRS at first. A path $Z(t) = (z(t), \theta(t))$ on SH gives a closed loop on the SRS if it satisfies the condition that there exist such an element $k \neq 1$ in $\text{S}\Gamma$ and a time interval T that

$$Z(t+T) = k(Z(t)) . \quad (48)$$

Since k is characterized by the two fixed points, (u, μ) and (v, ν) , the sign factor $\chi(k)$ and the norm function N_k (see Sect.2), the above condition gives a necessary condition,

$$\begin{aligned} \frac{z(t+T) - u - \theta(t+T)\mu}{z(t+T) - v - \theta(t+T)\nu} &= N_k \frac{z(t) - u - \theta(t)\mu}{z(t) - v - \theta(t)\nu} , \\ \frac{\theta(t+T) + \frac{\nu-\mu}{u-v}z(t+T) + \frac{v\mu-u\nu}{u-v}}{z(t+T) - v - \theta(t+T)\nu} &= \chi(k) N_k^{1/2} \frac{\theta(t) + \frac{\nu-\mu}{u-v}z(t) + \frac{v\mu-u\nu}{u-v}}{z(t) - v - \theta(t)\nu} . \end{aligned} \quad (49)$$

We find that the classical motions $Z^{(\text{II})}(t)$ (45) and $Z^{(\text{III})}(t)$ (46) do not satisfy the above condition (49) and only the motions $Z^{(\text{I})}(t)$ (44) with the parameters having values,^{*,†}

$$c_1 = \frac{v-u}{2}, \quad c_2 = \frac{u+v}{2}, \quad \xi_2 = \frac{\nu}{2}, \quad \xi_4 = \frac{\mu}{2}, \quad \xi_1 = \xi_3 = 0, \quad (50)$$

satisfy the original condition (48) and the time interval T is

$$T = \frac{\log N_k}{\omega}. \quad (51)$$

The path $Z^{(\text{I})}(t)$ with (50), which we denote $Z_k(t)$ associated with the element k ,

^{*} We may interchange (u, μ) and (v, ν) in the following equations, which corresponds to changing the sign of ω , or the direction of motion.

[†] The last two conditions $\xi_1 = \xi_3 = 0$ are not necessary for $Z^{(\text{I})}(t)$ to satisfy the condition (49) if $a = 2$ and $n > 0$, however, they are necessary to satisfy the condition (48).

is the geodesic curve connecting the two fixed points of the element $k \neq 1$ in $\text{S}\Gamma$

$$\begin{aligned} Z_k(t \rightarrow +\infty) &\longrightarrow (v, \nu), \\ Z_k(t \rightarrow -\infty) &\longrightarrow (u, \mu), \quad \omega > 0. \end{aligned} \quad (52)$$

A segment $[Z_k(t), Z_k(t+T)]$ of the geodesic curve becomes a closed loop on the SRS and the length of the loop $l(k)$ is given by

$$l(k) \equiv d(Z_k(t), Z_k(t+T)) = d(Z_k(t), k(Z_k(t))) = \log N_k, \quad (53)$$

which in fact depends only on the element $k \in \text{S}\Gamma$. Equation (53) yields

$$l(k^n) = |n| l(k). \quad (54)$$

The geodesic segment $[Z_k(t), Z_k(t+nT)]$ becomes a closed loop lying $|n|$ -fold exactly on the closed loop coming from the segment $[Z_k(t), Z_k(t+T)]$. So $[Z_k(t), Z_k(t+nT)]$ and $[Z_k(t), Z_k(t+T)]$ determine the same primitive periodic orbit on the SRS, and we conclude that two elements k^m and $k^n \neq 1$ (m, n : integers) in $\text{S}\Gamma$ are associated with the same primitive periodic orbit on the SRS. Furthermore, due to $\text{SPL}(2, \mathbb{R})$ -invariance of $d(q_1, q_2)$, we get

$$l(k) = d(gZ_k(t), gZ_k(t+T)) = d(gZ_k(t), g k g^{-1}(gZ_k(t))), \quad g \in \text{S}\Gamma. \quad (55)$$

This implies that $gZ_k(t)$ is the geodesic curve connecting the two fixed points of the element $g k g^{-1} \in \text{S}\Gamma$. Since $gZ_k(t)$ and $Z_k(t)$ become the same trajectory on the SRS, we conclude that every geodesic curve connecting the fixed points of each element of $\text{Conj}(\text{S}\Gamma)$ becomes the same orbit on the SRS. Thus we find that each pair $(p, p^{-1}) \in \text{Prim}(\text{S}\Gamma)$ is associated with a primitive periodic orbit on the SRS and its length is given by

$$l(p) = \log N_p = \log N_{p^{-1}}, \quad (56)$$

where N_p is the norm function associated with p . Conversely any periodic orbit on the SRS can be lifted to a geodesic segment $[Z(t), k(Z(t))]$ on SH with some element k

1 in $\Sigma\Gamma$. Since there exists a unique geodesic curve connecting the two points $Z(t)$ and $k(Z(t)) = Z(t + T)$, the geodesic curve is in fact a solution $Z^{(1)}(t)$ connecting the two fixed points of k . Then we conclude that there exists a one-to-one correspondence between primitive periodic orbits on the SRS and pairs of inconjugate primitive elements (p, p^{-1}) . Any geodesic curve $Z^{(1)}(t)$ not connecting two fixed points of any element in $\Sigma\Gamma$ becomes a nonperiodic orbit on the SRS and such geodesic curves are *dense* on SH. Hence the classical motion on the SRS is chaotic.

Signals of Chaos:

1. the lagrangian (26) is $\text{SPL}(2, \mathbb{R})$ -invariant, however, after projecting out onto the SRS, we find that the symmetry generators on SH become no longer those on the SRS and only two Grassmann even quantities are conserved ones, which are the Hamiltonian H and a nilpotent quantities $H^{(2)}$ essentially corresponding to $E^\theta E^{\bar{\theta}*}$. The fact that there are two kind of conserved quantities has been already presented in constructing the Lagrangian which consists of two $\text{SPL}(2, \mathbb{R})$ -invariant pieces. However, the dimension of the hyper surface determined by $H = E$ and $H^{(2)} = E^{(2)}$ ($E, E^{(2)}$: constants) in the total super space becomes less by one bosonic degree than that of total space according to the (super) implicit function theorem [5].
2. we will study the Anosov property [15] which describes the behavior of the initially neighboring trajectories at large times and is suitable to study the strongly chaotic systems [16]. Let us take two geodesic curves $Z^{(1)}(t)$ ($\omega > 0, t_0 = 0$) with the conditions (50) and another one with

$$c_1 = \frac{v + \delta v - u}{2}, \quad c_2 = \frac{u + v + \delta v}{2}, \quad \xi_2 = \frac{\nu + \delta \nu}{2}, \quad \xi_4 = \frac{\mu}{2}, \quad \xi_1 = \xi_3 = 0. \quad (57)$$

These two trajectories start from the same point (u, μ) at $t \rightarrow -\infty$, how-

ever, arrive at slightly different points (v, ν) and $(v + \delta v, \nu + \delta \nu)$ when $t \rightarrow \infty$. At $t = 0$ the value of separation is

$$d_{t=0} \sim \delta v + \frac{\mu + \nu}{2} \delta \mu, \quad (58)$$

However as $t \rightarrow \infty$ the trajectories separate exponentially,

$$d_{t \rightarrow \infty} \sim (\delta v + \nu \delta \nu) e^{\omega t}. \quad (59)$$

The velocity ω is the Liapunov exponent. This implies that trajectories are unstable, which is characteristic of classical chaos.

3. we comment on the Kolmogorov-Sinai entropy h [17] which, roughly speaking, measures unpredictability of the motions. This number comes out from the asymptotic formula for the counting function of primitive orbits of period $T(p) \leq T$,

$$\#\{p, T(p) \leq T\} \sim \frac{e^{hT}}{hT}, \quad T \rightarrow +\infty, \quad (60)$$

which indicates the exponential proliferation of the periodic orbits. From (51) and (56), this formula yields

$$\#\{p, l(p) \leq x\} \sim \frac{e^{\alpha x}}{\alpha x}, \quad x \rightarrow \infty \quad \text{with} \quad \alpha = \frac{h}{\omega}. \quad (61)$$

The asymptotic formula (61) will be discussed in the quantum mechanical framework in Sect. 6.

★ The explicit form of $H^{(2)}$ is

$$H^{(2)} = Y(\theta \bar{\theta} p_z p_{\bar{z}} - \theta p_z p_{\bar{\theta}} - \bar{\theta} p_{\bar{z}} p_{\theta} - p_{\theta} p_{\bar{\theta}}),$$

where $p_A (A = z, \bar{z}, \theta, \bar{\theta})$ denotes the canonical momentum.

4. Quantum Mechanics

In this section we develop the quantum theory for a particle moving on the SRS. As was the conventional Hadamard model, we cannot *quantize* the classical motion on the SRS because of its ergodicity. However there exists a well-defined quantum mechanics on the SRS where the Laplace-Beltrami operator acts as the quantum Hamiltonian. To construct the quantum mechanics on the SRS, we firstly develop the quantum mechanics on SH where the quantum Hamiltonian is also the Laplace-Beltrami operator. The quantum motion on the SRS is obtained by imposing the *periodic* boundary conditions upon the motion on SH just as the case of the classical motion. The quantum motion on SH is also expected to be integrable. In fact, as we will see, we can solve the Schrödinger equation explicitly and obtain the exact energy spectrum for the quantum motion on SH. A wave function on the SRS will be obtained by *folding* that on SH, however the function is quite complicated.

First we give the quantum Hamiltonian. Our Lagrangian (26) is nonlinear in a sense that g_{AB} are functions of supercoordinates. Omote and Sato [18] developed a procedure to construct Hamiltonian for a system with a (purely bosonic) nonlinear Lagrangian of a form $L_B = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ with having considered a symmetry as a guiding principle (see also [19]). We can follow their arguments paying attention to sign factors. We find the quantum Hamiltonian on SH,

$$H_Q = \frac{(-)^A}{2m} g^{-1/4} p_A g^{1/2} g^{AB} p_B g^{-1/4} , \quad (62)$$

where

$$g \equiv |\text{sdet } g_{AB}| = (4a^2 Y^2)^{-1} , \quad (63)$$

with the canonical commutation relations,

$$[p_A, q^B]_{\pm} = -i\hbar \delta_A^B . \quad (64)$$

The Hamiltonian H_Q is also $\text{SPL}(2, \mathbb{R})$ -invariant and hence is the Hamiltonian on SRS.

In the q -representation the coordinates q^A and the momenta p_A are given by

$$\begin{aligned} q^A &= q^A , \\ p_A &= -i\hbar g^{-1/4} \frac{\partial}{\partial q^A} g^{1/4} \equiv -i\hbar g^{-1/4} \partial_A g^{1/4} , \end{aligned} \quad (65)$$

so that H_Q is a super Laplace-Beltrami operator,

$$\begin{aligned} H_Q &= -\frac{\hbar^2}{2m} (-)^A g^{-1/2} \partial_A (g^{1/2} g^{AB} \partial_B) \\ &= -\frac{\hbar^2}{2m} \left[(2Y D \overline{D})^2 + \frac{1-a}{a} (2Y D \overline{D}) \right] \equiv \frac{\hbar^2}{2m} \triangle_{\text{SLB}} , \end{aligned} \quad (66)$$

where D is given in (5) and \overline{D} is its complex conjugate. The Hilbert space \mathcal{H} of our model is the space of superfunctions on SH with the inner product

$$\langle \Psi_1 | \Psi_2 \rangle \equiv \int d^4 q g^{1/2}(q) \langle \Psi_1 | q \rangle \langle q | \Psi_2 \rangle , \quad (67)$$

and H_Q is hermitian with respect to this product.

We will study the spectral properties of H_Q on SH. In order to do that we examine the eigenvalue problem of the operator \square_0 ,

$$\square_0 \equiv 2Y D \overline{D} . \quad (68)$$

The Grassmann even (odd) eigenfunction $e_{\Lambda}(\psi_{\Lambda})$, with eigenvalue $\lambda^B(\lambda^F)$,

$$\begin{aligned} \square_0 e_{\Lambda} &= \lambda^B e_{\Lambda} , \\ \square_0 \psi_{\Lambda} &= \lambda^F \psi_{\Lambda} , \end{aligned} \quad (69)$$

may be expanded as

$$\begin{aligned} e_{\Lambda} &= A_{\Lambda} + \theta \bar{\theta} B_{\Lambda} , \\ \psi_{\Lambda} &= \frac{1}{\sqrt{y}} (\theta \rho_{\Lambda} + \bar{\theta} \varphi_{\Lambda}) , \end{aligned} \quad (70)$$

where $\{A_{\Lambda}, B_{\Lambda}\}$ and $\{\rho_{\Lambda}, \varphi_{\Lambda}\}$ are functions of Grassmann even coordinates z

$x + iy$ and \bar{z} . Eqs. (69) and (70) yield

$$\begin{aligned} B_\Lambda &= \frac{\lambda^B}{2y} A_\Lambda , \\ \left\{ y^2 (\partial_x^2 + \partial_y^2) - \lambda^B (\lambda^B - 1) \right\} A_\Lambda &= 0 . \\ \varphi_\Lambda &= -\frac{2y\sqrt{y}}{\lambda^F} \partial_{\bar{z}} \left(\frac{\rho_\Lambda}{\sqrt{y}} \right) , \\ \left\{ y^2 (\partial_x^2 + \partial_y^2) - iy \partial_x - ((\lambda^F)^2 - \frac{1}{4}) \right\} \rho_\Lambda &= 0 , \end{aligned} \quad (71)$$

and the above differential equations can be solved as [★],

$$\begin{aligned} A_\Lambda &= C_{\lambda^B, k} e^{ikx} \sqrt{y} K_{\lambda^B - \frac{1}{2}}(|k|y) , \\ \rho_\Lambda &= C_{\lambda^F, k} e^{ikx} W_{\frac{\sigma_k}{2}, \lambda^F}(2|k|y) , \end{aligned} \quad (72)$$

where $\sigma_k \equiv \text{sign}(k)$, $k \neq 0$ and K_μ and $W_{\kappa, \mu}$ are a modified Bessel function and a Whittaker function, respectively. The normalization constants $C_{\lambda^B, k}$ and $C_{\lambda^F, k}$ are determined as follows. Since SH is noncompact and the spectrum is continuous, the normalization condition should be

$$\begin{aligned} \langle e_\Lambda | e_{\Lambda'} \rangle &\propto \delta(\lambda^B - (\lambda^B)') , \\ \langle \psi_\Lambda | \psi_{\Lambda'} \rangle &\propto \delta(\lambda^F - (\lambda^F)') , \end{aligned} \quad (73)$$

The above condition determines the regions of the eigenvalues [11],

$$\begin{aligned} \lambda^B &= \frac{1}{2} + ip , \\ \lambda^F &= c + ip , \end{aligned} \quad (74)$$

where $p \in (-\infty, +\infty)$ and $\{c: \text{real constant}, |c| \leq \frac{1}{2}\}$. And the normalized eigen-

★ When $k = 0$, $A_\Lambda \sim y^\lambda$ or $y^{1-\lambda}$ and $\rho_\Lambda \sim y^{1/2 \pm \lambda}$. Then they are unnormalizable on SH.

functions are given by,

$$\begin{aligned} e_{p,k}(Z) &= \left(\frac{2ia \sinh \pi p}{\pi^3} \right)^{1/2} \left(1 + \frac{1+2ip}{4y} \theta \bar{\theta} \right) e^{ikx} \sqrt{y} K_{ip}(|k|y) , \\ \psi_{p,k}^c(Z) &= \left(\frac{a \cos[\pi(c+ip)]}{2\pi^2 k(c+ip)^{\sigma_k-1}} \right)^{1/2} \frac{1}{\sqrt{y}} e^{ikx} \\ &\quad \times \left\{ \theta W_{\frac{\sigma_k}{2}, c+ip}(2|k|y) + i(c+ip)^{\sigma_k} \bar{\theta} W_{-\frac{\sigma_k}{2}, c+ip}(2|k|y) \right\} , \end{aligned} \quad (75)$$

with

$$\begin{aligned} \langle e_{q,l} | e_{p,k} \rangle &= \delta(p+q) \delta(k-l) , \\ \langle \psi_{q,l}^c | \psi_{p,k}^c \rangle &= \delta(k-l) \delta(p+q) . \end{aligned} \quad (76)$$

The eigenvalues of H_Q are

$$\begin{aligned} E_{p,k}^B &= \frac{\hbar^2}{2m} \left\{ \left(\frac{1-a}{2a} \right)^2 + \left(p - \frac{i}{2a} \right)^2 \right\} \equiv \frac{\hbar^2}{2m} \gamma^B(p) , \\ E_{p,k,c}^F &= \frac{\hbar^2}{2m} \left\{ \left(\frac{1-a}{2a} \right)^2 + \left(p - ic - i \frac{1-a}{2a} \right)^2 \right\} \equiv \frac{\hbar^2}{2m} \gamma_c^F(p) . \end{aligned} \quad (77)$$

Although H_Q is a hermite operator, the eigenvalue is complex. This is because the space of eigenstates contains isovectors [20] as is seen in (76). Notice that except when $c = \frac{1}{2}$ or $\frac{a-2}{2a}$ with $a \geq 1$, the energy spectra of the Grassmann even states and the odd ones do not coincide with each other,

$$\{E_{p,k}^B\} \neq \{E_{q,l,c}^F\} , \quad c \neq \frac{1}{2} \text{ and } \frac{a-2}{2a} \quad (a \geq 1) . \quad (78)$$

A set of eigenfunctions for each c , $\{e_{p,k}, \psi_{p,k}^c\}$, do satisfy the completeness relation,

$$\begin{aligned} \int_{-\infty}^{\infty} dp dk [e_{p,k}(q_2) \bar{e}_{-p,k}(q_1) + \psi_{p,k}^c(q_2) \bar{\psi}_{-p,k}^c(q_1)] \\ = [g(q_1)g(q_2)]^{-1/4} \delta(q_1 - q_2) . \end{aligned} \quad (79)$$

For each c , we have a Hilbert space for the Grassmann odd states \mathcal{H}_c^F , and hence

the total Hilbert space is

$$\mathcal{H}_c = \mathcal{H}^B \oplus \mathcal{H}_c^F, \quad (80)$$

where \mathcal{H}^B is the Hilbert space for the Grassmann even states.

We now turn our attention to the eigenfunctions on the SRS or F, a fundamental domain of $S\Gamma$. The $S\Gamma$ -invariance of the eigenfunction Ψ imposes the *periodic* boundary condition. The action of $g \in S\Gamma$ on Ψ reads

$$\Psi'(q) = [g\Psi](q) = \Psi(g^{-1}q), \quad q \in \text{SH}. \quad (81)$$

Hence, the periodicity is expressed as

$$\Psi(g^{-1}q) = \Psi(q) \quad \text{for all } g \in S\Gamma. \quad (82)$$

The spectrum of the operator Δ_{SLB} on SH, $\{\gamma^B(p)\}$ and $\{\gamma^F(p)\}$ in (77), will become discrete on the SRS. We then write the discrete spectrum of Δ_{SLB} as

$$\begin{aligned} \{\gamma_n^B\} & \quad (n = 0, 1, 2, \dots) \quad \text{for Grassmann even state,} \\ \{\gamma_n^F\} & \quad (n = 0, 1, 2, \dots) \quad \text{for Grassmann odd state,} \end{aligned} \quad (83)$$

and that of the operator \square_0 in (68) as

$$\begin{aligned} \{\lambda_n^B\} & \quad (n = 0, 1, 2, \dots) \quad \text{for Grassmann even state,} \\ \{\lambda_n^F\} & \quad (n = 0, 1, 2, \dots) \quad \text{for Grassmann odd state,} \end{aligned} \quad (84)$$

where

$$\gamma_n^{B(F)} = -(\lambda_n^{B(F)})^2 - \left(\frac{1-a}{a}\right)\lambda_n^{B(F)}. \quad (85)$$

However, because the periodic condition is complicated it is very difficult to see the spectrum on the SRS explicitly. We comment on the ground state. The

Grassmann even ground state is given by a constant function. It is a trivial periodic function with the normalization

$$\int_F dV(\text{const.}) < \infty, \quad (86)$$

and has the energy $\lambda_0^B = 0$ ($\gamma_0^B = 0$).

We now construct the kernel function on SH. The kernel function is given by

$$\begin{aligned} K(q_1, q_2; t) & \equiv \langle q_2 | e^{-\frac{it}{\hbar} H_Q} | q_1 \rangle, \\ & = \int_{-\infty}^{\infty} dp dk \left\{ e^{-\frac{it}{\hbar} E_{p,k}^B} \langle q_2 | e_{p,k} \rangle \langle e_{-p,k} | q_1 \rangle \right. \\ & \quad \left. + e^{-\frac{it}{\hbar} E_{p,k,c}^F} \langle q_2 | \psi_{p,k}^c \rangle \langle \psi_{-p,k}^c | q_1 \rangle \right\} \\ & \equiv K(q_1, q_2 | \tau), \end{aligned} \quad (87)$$

where

$$\tau \equiv \frac{i\hbar}{2m} t. \quad (88)$$

Plugging (75) into (87), we get [10, 11],

$$K(q_1, q_2 | \tau) = K^{(0)}(l; \tau) + r(q_1, q_2) K^{(1)}(l; \tau), \quad (89)$$

where $l = l(q_1, q_2)$ is given in (42) and,

$$\begin{aligned} K^{(0)}(l; \tau) & = \frac{-2a}{\pi\sqrt{2\pi\tau}} e^{-\left(\frac{1-a}{2a}\right)^2 \tau} \int_l^{\infty} db \frac{e^{-\frac{b^2}{4\tau}} \sinh \frac{b}{2a}}{(\cosh b - \cosh l)^{1/2}}, \\ K^{(1)}(l; \tau) & = \frac{-2a}{\pi\sqrt{2\pi\tau}} e^{-\left(\frac{1-a}{2a}\right)^2 \tau} \int_l^{\infty} db \frac{1}{(\cosh b - \cosh l)^{1/2}} \\ & \quad \times \left[(\cosh l - 1) \frac{d}{db} \left(e^{-\frac{b^2}{4\tau}} \frac{\sinh \frac{b}{2a}}{\sinh b} \right) \right] \end{aligned} \quad (90)$$

$$+ e^{-\frac{b^2}{4\tau}} \left(\frac{b}{2\tau} \cosh \frac{b}{2a} + \frac{a-1}{2a} \sinh \frac{b}{2a} \right) \Big] . \quad (91)$$

So the time development for a wave function on SH is given by

$$\Psi(q, t) = \int_{\text{SH}} dV(q') K(q, q'; t - t') \Psi(q', t') . \quad (92)$$

As for the wave function on the SRS, we should have

$$\Psi_{SRS}(q, t) = \int_{SRS} dV(q') K_{SRS}(q, q'; t - t') \Psi_{SRS}(q', t') \quad (93)$$

The periodicity of Ψ_{SRS} implies that a kernel K_{SRS} on the SRS is written as

$$K_{SRS}(q_1, q_2 | \tau) = \sum_{g \in \text{S}\Gamma} K(q_1, g(q_2) | \tau). \quad (94)$$

5. Trace Formula and Zeta Function

In the preceding sections, we have eventually solved the quantum mechanics on SH and obtained the heat kernel on SH, which yields that on the SRS. Here in this section, we will concentrate on the quantum energy spectrum on the SRS. As we have seen that the spectrum on the SRS was too complicated, it is quite difficult to estimate it explicitly. However, in the conventional case, that is, in the case of a particle on the Riemann surface of genus $g \geq 2$, the energy spectrum is related to the length spectrum through the Selberg trace formula [2]. We may expect that a similar relation will exist in our model.

First we present a formula of supertrace of a function $G_{\text{SRS}}(q_1, q_2)$ on $\text{SH}/\text{S}\Gamma$ which is made out of a $\text{SPL}(2, \mathbb{R})$ -invariant function $G(q_1, q_2)$ on $\text{SH} \times \text{SH}$

$$G_{\text{SRS}}(q_1, q_2) = \sum_{g \in \text{S}\Gamma} G(q_1, g(q_2)) , \quad (95)$$

with

$$G(q_1, q_2) = \Phi(l(q_1, q_2)) + r(q_1, q_2) \Psi(l(q_1, q_2)) , \quad (96)$$

where Φ and Ψ are some functions and l and r are given in (42) and (39), respectively. We find that the supertrace of G_{SRS} defined by,

$$\text{str } G_{\text{SRS}} = \int_{\text{F}} dV(q) G_{\text{SRS}}(q, q) = \sum_{g \in \text{S}\Gamma} \int_{\text{F}} dV(q) G(q, g(q)) , \quad (97)$$

where F is a fundamental domain of $\text{S}\Gamma$, is calculated as

$$\text{str } G_{\text{SRS}} = \text{Area}(\text{SRS}) \Phi(0) + \sum_{p \in \text{Prim}(\text{S}\Gamma)} \sum_{n=1}^{\infty} \int_{\text{F}_p} dV G(q, p^n(q)) , \quad (98)$$

where use has been made of a formula,

$$\sum_{g \in \text{S}\Gamma} f(g) = f(1) + \sum_{p \in \text{Prim}(\text{S}\Gamma)} \sum_{n=1}^{\infty} \sum_{g \in \text{S}\Gamma/Z(p)} f(gp^n g^{-1}) . \quad (99)$$

and F_p is a fundamental domain for the centralizer of p , $Z(p)$,

$$\text{F}_p = \bigcup_{g \in \text{S}\Gamma/Z(p)} g^{-1} \text{F} . \quad (100)$$

We can assume that p is a magnification with a matrix $A_p = \text{diag}(\chi(p) N_p^{1/2})$

$\chi(p)N_p^{-1/2}, 1)$ (see (14)) and we can choose a convenient domain for F_p [14],

$$\int_{F_p} dV \Rightarrow \int_1^{N_p} dy \int_{-\infty}^{\infty} dx \int d\theta d\bar{\theta} \frac{1}{2ay + a\theta\bar{\theta}} . \quad (101)$$

Then we finally get

$$\begin{aligned} \text{str } G_{\text{SRS}} &= \frac{\pi(g-1)}{a} \Phi(0) \\ &- \sum_{p \in \text{Prim}(\text{SF})} \sum_{n=1}^{\infty} \frac{l(p)}{2a\sqrt{\cosh l(p^n) - 1}} \\ &\times \left\{ \left(1 - \chi(p^n) \cosh \frac{l(p^n)}{2} \right) \int_{l(p^n)}^{\infty} ds \frac{\sinh s}{(\cosh b - \cosh l(p^n))^{1/2}} \Psi(s) \right. \\ &\quad \left. + (1 - \cosh l(p^n)) \int_{l(p^n)}^{\infty} ds \frac{1}{(\cosh b - \cosh l(p^n))^{1/2}} \frac{d\Phi(s)}{ds} \right\} , \end{aligned} \quad (102)$$

where g is the genus of the SRS and

$$l(p^n) = |n|l(p) = |n| \log N_p . \quad (103)$$

Now we apply the above formula to the heat kernel on the SRS (94) which can be written as

$$K_{\text{SRS}}(q_1, q_2 | \tau) = \sum_{g \in \text{SF}} \langle q_1 | e^{-\tau \Delta_{\text{SLB}}} | g(q_2) \rangle . \quad (104)$$

Then we get

$$\text{str } K_{\text{SRS}} = \text{str} (e^{-\tau \Delta_{\text{SLB}}}) = \sum_{n=0}^{\infty} \left(e^{-\tau \gamma_n^B} - e^{-\tau \gamma_n^F} \right) . \quad (105)$$

Plugging $K^{(0)}$ (90) and $K^{(1)}$ (91) respectively into Φ and Ψ in (102) and integrat-

ing with respect to s , we finally obtain a superanalog of Selberg trace formula [11]

$$\sum_{n=0}^{\infty} \left(e^{-\tau \gamma_n^B} - e^{-\tau \gamma_n^F} \right) = A(\tau) + \Theta(\tau) , \quad (106)$$

where

$$A(\tau) = \frac{1}{\sqrt{\pi\tau}} (1-g) e^{-(\frac{1-a}{2a})^2 \tau} \int_0^{\infty} db e^{-\frac{b^2}{4\tau}} \frac{\sinh \frac{b}{2a}}{\sinh \frac{b}{2}} , \quad (107)$$

$$\Theta(\tau) = \frac{1}{4\sqrt{\pi\tau}} \sum_{p \in \text{Prim}(\text{SF})} \sum_{n=1}^{\infty} \text{str} (K(p^n | a)) \frac{l(p)}{\sinh \frac{l(p^n)}{2}} e^{-\frac{l^2(p^n)}{4\tau} - (\frac{1-a}{2a})^2 \tau} , \quad (108)$$

$$K(p | a) = \text{diag} \left(e^{\frac{l(p)}{2a}}, e^{-\frac{l(p)}{2a}}, \chi(p) e^{\frac{1-a}{2a} l(p)}, \chi(p) e^{-\frac{1-a}{2a} l(p)} \right) . \quad (109)$$

$A(\tau)$ is the contribution of the “trivial motion” on the SRS (zero length term) and for $\tau \rightarrow 0$ it can be expanded into a positive power series,

$$A(\tau) \sim -\frac{\text{Area}(\text{SRS})}{\pi} (b_0 + b_1 \tau + b_2 \tau^2 + \dots) , \quad (110)$$

with

$$\begin{aligned} b_0 &= 1 , \\ b_1 &= \frac{(a-1)(1-2a)}{6a^2} , \\ b_2 &= \frac{(a-1)(2a-1)(2a^2-2a+1)}{60a^4} , \\ &\vdots \end{aligned} \quad (111)$$

This series approximates “ $\text{str} (e^{-\tau \Delta_{\text{SLB}}})$ ” up to an exponentially small error. On the other hand, $\Theta(\tau)$ is the contribution from the periodic motions on the SRS and consistent with the semiclassical approximation [9].

Let us introduce a zeta function $Z(s|a)$ [10] with one parameter a associated with our model. The function is defined by,

$$Z(s|a) \equiv \prod_{p \in \text{Prim}(S\Gamma)} \prod_{n=0}^{\infty} \text{sdet} \left(1 - K(p|a) e^{-(s+n)l(p)} \right) , \quad (112)$$

and we see that the zeta function is related to the trace formula [11],

$$\frac{d}{ds} \log Z(s|a) = (2s-1) \int_0^{\infty} dt e^{-(s-\frac{1}{2})^2 t + (\frac{1-a}{2a})^2 t} \Theta(t) . \quad (113)$$

We point out that the zero-points and the poles of $Z(s|a)$ give directly the eigenvalues of Δ_{SLB} (energy spectrum) on the SRS. More precisely, the zero-points give the eigenvalues of the Grassmann even functions and the poles give those of the odd functions. Using the trace formula, we find

$$\begin{aligned} \frac{d}{ds} \log Z(s|a) &= (2s-1) \int_0^{\infty} dt e^{-(s-\frac{1}{2})^2 t + (\frac{1-a}{2a})^2 t} \left\{ \sum_{n=0}^{\infty} (e^{-t\gamma_n^B} - e^{-t\gamma_n^F}) - A(t) \right\} \\ &= (2s-1) \sum_{n=0}^{\infty} \left[\frac{1}{(s-\frac{1}{2})^2 - (\frac{1-a}{2a})^2 + \gamma_n^B} - \frac{1}{(s-\frac{1}{2})^2 - (\frac{1-a}{2a})^2 + \gamma_n^F} \right] \\ &\quad + 2(g-1) \sum_{n=0}^{\infty} \left(\frac{1}{s+n-\frac{1}{2a}} - \frac{1}{s+n+\frac{1}{2a}} \right) . \end{aligned} \quad (114)$$

Note that the last term in (114) becomes a finite sum when $|a|^{-1} = m$ (m : positive integers),

$$2(g-1) \text{sign}(a) \sum_{n=0}^{m-1} \frac{1}{s+n-\frac{m}{2}} . \quad (115)$$

This formula implies that $Z(s|a)$ has a meromorphic continuation onto the whole

complex plane \mathbb{C} . The zero-points (poles) of order 1 exist at

$$s = \frac{1}{2} \pm \sqrt{\left(\frac{1-a}{2a}\right)^2 - \gamma_n^{B(F)}} . \quad (116)$$

Other trivial zero-points (ZP) and poles (P) exist respectively at,

I. $a^{-1} \neq m$ (m : positive integers);

$$\begin{aligned} \text{ZP : } \quad s &= -n + \frac{1}{2a} , \\ \text{P : } \quad s &= -n - \frac{1}{2a} , \quad (n = 0, 1, 2, \dots) , \end{aligned} \quad (117)$$

II. $a^{-1} = m$;

$$\begin{aligned} \text{ZP : } \quad s &= -n + \frac{m}{2} , \\ \text{P : } \quad &\text{none} , \quad (n = 0, 1, \dots, m-1) , \end{aligned} \quad (118)$$

III. $a^{-1} = -m$;

$$\begin{aligned} \text{ZP : } \quad &\text{none} , \\ \text{P : } \quad s &= -n - \frac{m}{2} , \quad (n = 0, 1, \dots, m-1) , \end{aligned} \quad (119)$$

where the order of each ZP and P is $2(g-1)$.

6. Classical Chaos from the Trace Formula

Our aim in this section is to discuss the exponential growth of the counting function $\Pi(x)$ for the lengths of primitive periodic orbits in (61),

$$\begin{aligned} \Pi(x) &= \#\{p, l(p) \leq x\} \\ &\sim \frac{e^{\alpha x}}{\alpha x}, \quad \text{for } x \rightarrow \infty. \end{aligned} \quad (120)$$

For this purpose, we prefer the general Selberg supertrace formula, established in Refs.[14, 21]. Specifically, if h is a test function with the properties

- (i) $h(\frac{1}{2} + ip) \in \mathbb{C}^\infty(\mathbb{R})$
- (ii) $h(\frac{1}{2} + ip) \sim O(\frac{1}{p^2})$ for $p \rightarrow \pm\infty$
- (iii) $h(\frac{1}{2} + ip)$ is holomorphic in the strip $|\text{Im } p| \leq 1 + \epsilon, \epsilon > 0$,

then the supertrace formula on the SRS with genus $g \geq 2$ is given by[★] [14],

$$\begin{aligned} &\sum_{n=0}^{\infty} [h(\lambda_n^B) - h(\lambda_n^F)] \\ &= (1 - g) \int_0^{\infty} \frac{g(u) - g(-u)}{\sinh \frac{u}{2}} du \\ &\quad + \sum_{p \in \text{Prim}(\text{S}\Gamma)} \sum_{n=1}^{\infty} \frac{l(p)}{2 \sinh \frac{nl(p)}{2}} \left[g(nl(p)) + g(-nl(p)) \right. \\ &\quad \left. - \chi^n(p) \left(g(nl(p)) e^{\frac{-nl(p)}{2}} + g(-nl(p)) e^{\frac{nl(p)}{2}} \right) \right], \end{aligned} \quad (121)$$

where $\{\lambda_n^{B(F)}\}$ is the spectrum of the operator \square_0 (see (84)) and

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-iup} h\left(\frac{1}{2} + ip\right). \quad (122)$$

To get the information for $\Pi(x)$, it is convenient to choose a test function h so

★ The trace formula reduces to (106) if one chooses the test function h as $h(\lambda_n^{B(F)}) = e^{-\gamma_n^{B(F)} \tau}$, where $\gamma_n^{B(F)} = -(\lambda_n^{B(F)})^2 - \frac{1-a}{a} \lambda_n^{B(F)}$.

that the term proportional to $\chi^n(p)$ in (121) cancel, i.e.,

$$g(nl(p)) e^{\frac{-nl(p)}{2}} + g(-nl(p)) e^{\frac{nl(p)}{2}} = 0. \quad (123)$$

We take a function ($\text{Re } s > 1, \text{Re } \sigma > 1$)

$$h(\lambda) = 2\lambda \left(\frac{1}{s^2 - \lambda^2} - \frac{1}{\sigma^2 - \lambda^2} \right), \quad (124)$$

with the Fourier transform $g(u)$ given by [21],

$$g(u) = \text{sign}(u) e^{\frac{u}{2}} (e^{-s|u|} - e^{-\sigma|u|}). \quad (125)$$

Thus only $\chi(p)$ -independent term remains in the supertrace formula. Plugging (124) and (125) into (121), we have

$$\begin{aligned} &2 \sum_{n=0}^{\infty} \left\{ \frac{\lambda_n^B}{s^2 - (\lambda_n^B)^2} - \frac{\lambda_n^F}{s^2 - (\lambda_n^F)^2} - \frac{\lambda_n^B}{\sigma^2 - (\lambda_n^B)^2} + \frac{\lambda_n^F}{\sigma^2 - (\lambda_n^F)^2} \right\} \\ &= 4(g-1) \{ \Psi(1+s) + \Psi(s) - \Psi(1+\sigma) - \Psi(\sigma) \} \\ &\quad + \sum_{p \in \text{Prim}(\text{S}\Gamma)} l(p) \left(\frac{e^{-sl(p)}}{1 - e^{-sl(p)}} - \frac{e^{-\sigma l(p)}}{1 - e^{-\sigma l(p)}} \right) \end{aligned} \quad (126)$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$. Defining

$$F(s) = \sum_{p \in \text{Prim}(\text{S}\Gamma)} l(p) \frac{e^{-sl(p)}}{1 - e^{-sl(p)}}, \quad (127)$$

and using a formula,

$$\Psi(z) = -\gamma - \sum_{n=0}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n+1} \right), \quad (128)$$

we obtain

$$\begin{aligned}
F(s) - F(\sigma) &= 4(g-1) \left\{ \sum_{n=0}^{\infty} \frac{1}{1+s+n} + \sum_{n=0}^{\infty} \frac{1}{s+n} - \sum_{n=0}^{\infty} \frac{1}{1+\sigma+n} - \sum_{n=0}^{\infty} \frac{1}{\sigma+n} \right\} \\
&+ \sum_{n=0}^{\infty} \left\{ \frac{1}{s-\lambda_n^B} - \frac{1}{s+\lambda_n^B} - \frac{1}{s-\lambda_n^F} + \frac{1}{s+\lambda_n^F} \right\} \\
&- \sum_{n=0}^{\infty} \left\{ \frac{1}{\sigma-\lambda_n^B} - \frac{1}{\sigma+\lambda_n^B} - \frac{1}{\sigma-\lambda_n^F} + \frac{1}{\sigma+\lambda_n^F} \right\}.
\end{aligned} \tag{129}$$

Now we can read the analytic properties of $F(s)$ from the above trace formula^{*}:

- (i) $F(s)$ has a meromorphic continuation onto the whole plane \mathbb{C} ,
- (ii) $F(s)$ has poles in the following points

$$\begin{aligned}
s = 0, & \quad \text{residue} \quad 4(g-1), \\
s = -n \quad (n = 1, 2, \dots), & \quad \text{residue} \quad 8(g-1), \\
s = \pm\lambda_n^B \text{ and } \lambda_n^B \neq \lambda_n^F, & \quad \text{residue} \quad \pm 1, \\
s = \pm\lambda_n^F \text{ and } \lambda_n^B \neq \lambda_n^F, & \quad \text{residue} \quad \mp 1.
\end{aligned} \tag{130}$$

Now let us return to the asymptotic formula of $\Pi(x)$ in (120). We will see that the exponential coefficient α in the equation is determined by the pole of $F(s)$ corresponding to the real eigenvalue of \square_0 . First we consider the series,

$$F(k) = \sum_{p \in \text{Prim}(\text{SF})} l(p) \frac{e^{-kl(p)}}{1 - e^{kl(p)}} \quad \text{for real positive } k. \tag{131}$$

The convergence of the above series is under control of the balance between the proliferation of the number of the paths and the exponential damping factor $e^{-kl(p)}$. When the variable k moves to the small direction, the singularity of

^{*} We have assumed the convergence of the series(129).

$F(k)$, which is connected with the largest real eigenvalue through the trace formula (129), appears on the real positive axis. On the other hand, we expect that the eigenvalues of \square_0 locate on (see (74)),

$$\begin{aligned}
\text{Re}(\lambda_n^B) &= \frac{1}{2}, \\
\text{Re}(\lambda_n^F) &= c \quad (|c| \leq \frac{1}{2}).
\end{aligned} \tag{132}$$

Thus the largest positive eigenvalue λ_{max} should exist and it takes the value $\frac{1}{2}$ or $c(>0)$ if we assume its existence and ignore so called *small eigenvalues*, which we have no knowledge except for the ground energy $\lambda_0^B = 0$. In the second step, we relate the first singularity of $F(k)$ to the proliferation of periodic orbits. In fact, we have

$$\begin{aligned}
F(k) &\sim \sum_p l(p) e^{-kl(p)} \quad l \rightarrow \infty, \\
&\sim \int x e^{-kx} d\Pi(x) \quad x \rightarrow \infty.
\end{aligned} \tag{133}$$

This implies by the above arguments

$$\int x e^{-kx} d\Pi(x) \sim \frac{1}{k - \lambda_{max}}. \tag{134}$$

If we now perform the inverse Laplace transformation, we get

$$\Pi(x) \sim e^{\lambda_{max}x} / x, \tag{135}$$

and hence

$$\alpha = \lambda_{max}. \tag{136}$$

Here we stress the following points:

- (i) the asymptotic for $\Pi(x)$ has been determined from the real eigenvalue λ_{max} which is quite different in comparison to the dynamical system on the ordinary Riemann surface. In the latter model, the ground energy ($\lambda_0 = 0$) controls the exponential proliferation of the periodic orbits through the trace formula and leads to the asymptotic formula

$$\Pi(x) \sim e^x/x. \quad (137)$$

- (ii) Our derivation includes the delicate arguments so the rigorous proof should be developed. One way to the direction is to investigate the analytic properties of L -function defined by

$$L(s) = Z(s+1)/Z(s) , \quad (138)$$

$$Z(s) = \prod_{p \in \text{Prim}(\text{SF})} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)l(p)}\right) ,$$

Then the following theorem is known [22],

Theorem

If $L(s)$ satisfies the conditions:

- (I) $L(s)$ converges absolutely for $\text{Re}(s) > \alpha$,
- (II) $L(s)$ has a meromorphic continuation onto some region including $\text{Re}(s) > \alpha$,
- (III) $L(s)$ has no zero point on $\text{Re}(s) > \alpha$,
- (IV) $L(s)$ is holomorphic on $\text{Re}(s) > \alpha$ and has a simple pole at $s = \alpha$, then the asymptotic formula exists

$$\pi(x) \sim e^{\alpha x}/\alpha x . \quad (139)$$

Our previous arguments support the conditions (I)-(IV), however, the constant α has not been determined strictly due to *small eigenvalue* problems.

7. Moduli

The energy spectrum is controlled by the length spectrum, or equivalently the periodic orbits on SRS through the trace formula and the length of a periodic orbit is determined by the norm function of the corresponding SF -element. This permits us to think of the *moduli* for SRS which is the free parameters in SF . In the theory of Riemann surface, it is known that some lengths corresponding to $6g - 6$ primitive periodic orbits can be chosen as the moduli parameters. The similar situation may be expected in the case of SRS. In Sec. 2, we saw that the moduli of SRS is the $6g - 6$ Grassmann even and $4g - 4$ odd parameters and hence the lengths of $6g - 6$ should be used as the even moduli parameter. Unfortunately, we have no quantity for the odd-moduli, i.e., the mechanical observable is Grassmann even.

We first note that the length $l(k)$, $k \in \text{SF}$ is written as

$$l(k) = \log(k(S), S, U_k, V_k) \quad (140)$$

Here $U_k(V_k)$ is the repelling (attractive) fixed point of k (see (9) and (10)) and is an arbitrary point in $\mathbf{R}_s - \{U_k, V_k\}$. The Grassmann even bracket of 4-points (Z_1, Z_2, Z_3, Z_4) on $\text{SH} \cup \mathbf{R}_s$ is defined by

$$(Z_1, Z_2, Z_3, Z_4) = \frac{z_{13}z_{24}}{z_{14}z_{23}}, \quad (141)$$

where $Z_i = (z_i, \theta_i)$ and $z_{ij} = z_i - z_j - \theta_i\theta_j$. This is invariant under $\text{SPL}(2, \mathbb{R})$ and its body is actually the ordinary cross ratio invariant under the Möbius transformations. Next we introduce the Grassmann odd $\text{SPL}(2, \mathbb{R})$ invariant quantity. This is defined by the bracket of 3-points on $\text{SH} \cup \mathbf{R}_s$,

$$(Z_1, Z_2, Z_3) = \frac{\theta_{123}}{(z_{12}z_{23}z_{31})^{1/2}}, \quad (142)$$

where $\theta_{123} = \theta_1z_{23} + \theta_2z_{31} + \theta_3z_{12} + \theta_1\theta_2\theta_3$. The ordering of 3-points in the above formula is fixed (up to cyclic permutations) by demanding that $(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) > 0$.

$z_3)(z_3 - z_1) > 0$. Using this invariant, we can provide $4g-4$ odd moduli parameters. Let $\{A_i, B_i\}$ ($i = 1 \sim g$) be the generators of SF in Eq.(7), and $\{U_i^A, V_i^A\}$ and $\{U_i^B, V_i^B\}$ be the fixed points of the generators A_i and B_i , respectively. Then odd moduli parameters $\{\lambda(k)\}$ ($k = 1 \sim 4g-4$) are given by [23],

$$\begin{aligned} \{\lambda(k)\} = & \{(U_1^A, V_1^A, U_j^A), (U_1^A, V_1^A, V_j^A), \\ & (U_1^A, V_1^A, U_j^A), (V_1^A, V_1^A, V_j^A), \quad j = 2, 3, \dots, h\}. \end{aligned} \quad (143)$$

Since the condition (7) on the generators is invariant under conjugation, one may regard A_1 as diagonal, i.e., the fixed points of A_1 can be put to $\{(0, 0), (\infty, 0)\}$. Then the condition reveals that the parameters in B_1 is written by the parameters in $\{A_j, B_j\}$ ($j = 2, \dots, g$). The $\{\lambda(k)\}$ represent essentially all the odd parameters in the remaining generators, $\{A_j, B_j\}$ ($j = 2, \dots, g$). Thus $\{l(i), \lambda(j)\}$ ($i = 1, \dots, 6g-6$, $j = 1, \dots, 4g-4$) provide the moduli of SRS. These moduli parameters have the manifest $\text{SPL}(2, \mathbb{R})$ invariance by the construction and offer the good coordinates on the moduli space of SRS.

REFERENCES

- [1] J. Hadamard, J. Math. Pure Appl. **4** (1898) 27.
- [2] for a recent review, see N.L. Balazs, and A. Voros, Phys. Rep. **143** (1986) 109.
- [3] A. Selberg, J. Indian Math. Soc. **20** (1956) 47.
- [4] D.A. Hejhal, *Lecture Note in Mathematics*, Vol. **548**; Berlin, Heidelberg, New York: Springer 1976.
- [5] F.A. Berezin, *Introduction to Superalgebra*, Math. Phys. and Appl. Math. Vol. **9**, : D. Reidel Publishing 1987.
- [6] D. Friedan, *in the Proceedings of the Workshop on Unified Theories*, edited by D. Gross and M. Green, Shingapore: World Scientific 1986.
- [7] S. Matsumoto and Y. Yasui, Prog. Theor. Phys. **79** (1988) 1022.
- [8] S. Uehara and Y. Yasui, Phys. Lett. **202B** (1988) 530.
- [9] S. Uehara and Y. Yasui, J. Math. Phys. **29** (1988) 2486.
- [10] S. Matsumoto, S. Uehara and Y. Yasui, Phys. Lett. **134A** (1988) 81.
- [11] S. Matsumoto, S. Uehara and Y. Yasui, J. Math. Phys. **31** (1990) 476.
- [12] L. Crane and J.M. Rabin, Comm. Math. Phys. **100** (1985) 141; **113** (1988) 601.
- [13] P. Howe, J. Phys. **A12** (1979) 393.
- [14] A.M. Baranov, Yu.I. Manin, I.V. Frolov and A.S. Schwarz, Comm. Math. Phys. **111** (1987) 373.
- [15] D.V. Anosov, Proc. Steklov Inst. of Math. **90** (1967).
- [16] Ya.G. Sinai, *Introduction to Ergodic Theory*, Princeton University Press 1976.
- [17] Ya.V. Pesin, Dokl. Akad. Nauk SSSR **226** (1976) (Soviet Math. Dokl. **17** (1976) 196).
- [18] M. Omote and H. Sato, Prog. Theor. Phys. **47** (1972) 1367.
- [19] T. Kawai, Foundation of Phys. **5** (1975) 143.
- [20] K. Aoki, Comm. Math. Phys. **117** (1988) 405.
- [21] C. Grosche, Comm. Math. Phys. **133** (1990) 433; **151** (1993) 1.
- [22] T. Sunada, *Kihon-gun to Laplacian* Kinokuniya Suugaku Sensho 29; Kinokuniya (1988).
- [23] S. Uehara and Y. Yasui, Phys. Lett. **217B** (1989) 479; Comm. Math. Phys. **144** (1992) 53.